# Infinite number of pure equilibrium states, Parisi order parameter and the ultrametric topology: a simple mean field model



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# Infinite number of pure equilibrium states, Parisi order parameter and the ultrametric topology: a simple mean field model

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**Abstract.** We discuss and solve by standard methods a simple model with long-range random interaction. In this model we can rigorously define and explicitly work out many peculiar features already found in the Sherrington-Kirkpatrick model only by means of replica symmetry breaking and/or via numerical simulations.

## 1. Introduction

Recent results in the replica approach to the Sherrington-Kirkpatrick (SK) model suggest the existence and physical relevance of model systems that have (Mezard *et al* 1984)

(i) an infinite number of coexisting pure equilibrium states;

(ii) a hierarchical structure among these states typical of ultrametric spaces (see point (c) of § 2) and

(iii) an order parameter which is non-self-averaging, i.e. it shows a non-trivial dependence from the actual realisation of the couplings. Meanwhile other quantities (e.g. the free energy) always take the same value in spite of the randomness of the interaction.

In this paper we discuss a simple random model, where, for almost every sample, we can explicitly define and evaluate

(i) the free energy;

(ii) the set of pure states;

(iii) the overlapping between any two of these states and

(iv) the probability distribution of such an overlapping.

Meanwhile the free energy is independent from the sample and the probability distribution (iv) is shown to be a non-self-averaging quantity.

This allows us to illustrate a possible way to deal rigorously with infinite sets of pure states and to discuss the problems involved in the description of a non-self-averaging effect.

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The system is made up with $2K$ blocks of $M$ spins each. The Hamiltonian is the sum of $2K$ self-interaction terms plus a suitable long-range potential. We study the model in the limit where $M$ and $K$ both go to infinity.

The self-interaction is such that for $M \to \infty$ each isolated block has four independent equilibrium states. Before the long-range terms are switched on, the equilibrium states of the full system are $4^{2K}$ and the global properties of this set of states are, in some sense, trivial. In particular if we compare the local magnetisation in two of these states the value of the overlapping $q$, i.e.

$$q = \frac{1}{2KM} \sum_{\alpha=1}^{2KM} \langle \sigma_\alpha \rangle \langle \sigma_\alpha \rangle'$$

is typically zero.

The form of the interaction among the blocks has been chosen in such a way that the new equilibrium states are contained in the set of the above mentioned $4^{2K}$ states, but belong to a suitable subset where the overlapping is non-trivial and the ultrametric structure together with properties (iii) follow.

We also consider the action of an external magnetic field (see § 5): we show that, when $h$ is sufficiently small, the general picture stays the same.

In our model randomness is present at two different levels. There are local random couplings *à la* Luttinger (1976) responsible for the existence of the above mentioned four pure equilibrium states in each block and a second family of random couplings, associated with the scale length of the blocks, and that gives rise to the fluctuations of the order parameter. From the sketchy description given above the reader easily realises that the model is strictly '*ad hoc*'. The mechanisms that we assume only aim to reproduce a given set of results and possibly do not have much to do with those present in physical systems or even in the SK model. In spite of this, we think it is of some interest to present an example where it is possible to properly define and explicitly derive quantities and relations suggested by replica symmetry breaking arguments.

## 2. Description of the model and main results

Consider the one-dimensional lattice $\mathbb{Z}$. We suppose we are given three random fields: $(\xi_\alpha)_{\alpha \in \mathbb{Z}}$, $(\eta_\alpha)_{\alpha \in \mathbb{Z}}$ and $(\zeta_{2i})_{i \in \mathbb{Z}}$, where $\xi_\alpha$, $\eta_\alpha$ and $\zeta_{2i}$ are independent identically distributed random variables taking the value $+1$ and $-1$ with equal probabilities. We call $\mu_1$ the probability corresponding to $(\xi_\alpha, \eta_\alpha)_{\alpha \in \mathbb{Z}}$ and $\mu_2$ the one associated to $(\zeta_{2i})_{i \in \mathbb{Z}}$. Our model will describe an Ising-like system: to each lattice site $\alpha \in \mathbb{Z}$ is associated a spin $\sigma_\alpha$ taking values $\pm 1$ and the interaction will depend on the random field $(\xi_\alpha)$ $(\eta_\alpha)$ and $(\zeta_{2i})$.

Given an integer $M$, we call

$$\begin{aligned} m_i &= \frac{1}{M} \sum_{\alpha=iM+1}^{(i+1)M} \sigma_\alpha \\ S_i &= \frac{1}{M} \sum_{\alpha=iM+1}^{(i+1)M} \xi_\alpha \sigma_\sigma \\ t_i &= \frac{1}{M} \sum_{\alpha=iM+1}^{(i+1)M} \eta_\alpha \sigma_\alpha . \end{aligned} \tag{2.1}$$

and define in a volume $\Omega=[1, N]$ with $N=2KM$ the following Hamiltonian

$$H(\boldsymbol{\sigma}_\Omega)=-J_0M\sum_{i=1}^{2K}\left(\frac{S_i^2}{2}+\frac{t_i^2}{2}\right)+J_1M\sum_{l=1}^{K}\sum_{J=2l+1}^{2K}$$
$$\times\frac{[(t_{2l}-t_{2l-2})^4+S_{2l-2}^4]t_J^4}{(J-2l)^{1+\varepsilon}}+\frac{1}{\sqrt{K}}\sum_{i=1}^{K}\zeta_{2i}t_{2i} \qquad (2.2)$$

with $J_0$, $J_1$ and $\varepsilon$ positive.

Notice that the terms in the first sum on the right-hand side of (2.2) are very similar to those considered by Luttinger (1976) in his non-frustrated mean field model of a spin glass and our Hamiltonian can be regarded as describing a set of $2K$ such systems coupled by a non-translationally invariant potential *à la* Kac (Thompson 1972) with zero boundary conditions. We shall study the equilibrium statistical mechanics of this system for a given sample of the $\{\xi_\alpha\}$ $\{\eta_\alpha\}$ and $\{\zeta_{2i}\}$. We are interested in the limit $M$ and $K$ going to infinity at the same time, but in order to be more clear we shall take the limit $M\to\infty$ first.

We remark that in our model, whose ancestor is the Curie–Weiss model (see, for instance, Thompson 1972) the microscopical variable appears in the Hamiltonian only via some linear combination suitably normalised, i.e. $S_i$, $t_i$ (see (2.1)). Therefore only these variables, taking values on the internal $[-1, +1]$, are relevant and our equilibrium states will be probability measures on $([-1, +1]\times[-1, 1])^{\mathbb{Z}}$. We will show that $\mu_1$ almost everywhere.

(a) For any realisation of the $\zeta_{2i}$

$$F(\beta)=\lim_{\Omega\to\infty}\frac{1}{|\Omega|}\log\sum_{\{\sigma_\Omega\}}\exp(-\beta H|\boldsymbol{\sigma}_\Omega)) \qquad (2.3)$$

$$=\begin{cases}0 & \text{if } 0\leq\beta\leq\beta_c\equiv 1/J_0\\ \beta J_0 m^{*2}/2-\mathscr{E}(m^*) & \text{if } \beta>\beta_c\end{cases} \qquad (2.4)$$

where $\mathscr{E}(x)\equiv\frac{1}{2}(1+x)\log(1+x)+\frac{1}{2}(1-x)\log(1-x)$ is the leading term in $|\Omega|$ of $\log\binom{|\Omega|}{\frac{1}{4}(1+x)|\Omega|}$ and $m^*$ is the positive solution of the equation

$$m^*=\tanh\beta J_0m^*.$$

(b) For all $\beta\leq\beta_c$ the equilibrium state is unique and the associated probability distribution has support on the configuration $S_i=0$, $t_i=0$, $\forall i\in\mathbb{Z}$.

For all $\beta>\beta_c$ the system exhibits an infinity of pure equilibrium states, each being characterised by a suitable sequence of $0, \pm m^*$ for the values of the $S_i$ and the $t_i$ (see figure 1). To be more precise, in the finite $K$ case, our Gibbs distribution converges in the limit $M\to\infty$ to a convex combination of $2K\ 2^{2K}$ Dirac measures. We shall call $N_K$ the set of the allowed sequences (see (3.14)) where these measures are concentrated.

**Figure 1.** Equilibrium states associated to the instanton $(J, +m^*)$ where $t=m^*$, $s=0$ for the blocks ▨, $t=0$, $s=\pm m^*$ for the blocks □, $t=\pm m^*$, $s=\pm m^*$ for the blocks ⊟.

Given an element $A$ in $N_K$ we call $S_i^A$, $t_i^A$, $i=1,\ldots,2K$ the values of the $S_i$ and $t_i$ in the sequence $A$.

(c) To analyse the structure of this set of states, in the limit of large $K$, consider, following Parisi (1983), two identical copies of our system. The function

$$g_{M,K}(y)=\sum_{\{\sigma_\Omega,\sigma'_\Omega\}}\exp\left(-\beta[H(\sigma_\Omega)+H(\sigma'_\Omega)]+\frac{y}{2MK}\sum_{\alpha=1}^{2MK}\sigma_\alpha\sigma'_\alpha\right)$$
$$\times\left(\sum_{\{\sigma_\Omega,\sigma'_\Omega\}}\exp\{-\beta[H(\sigma_\Omega)+H(\sigma'_\Omega]\}\right)^{-1} \tag{2.5}$$

is the Laplace transform of the probability distribution of overlapping $((1/2MK)\sum_{\alpha=1}^{2MK}\sigma_\alpha\sigma'_\alpha)$ between the spins of the two copies. Performing the limit $M\to\infty$ we obtain (see the appendix)

$$\lim_{M\to\infty} g_{M,K}(y)=\sum_{\mathrm{A,B}\in N_K}\exp\left(\frac{\beta}{\sqrt{K}}\sum_{i=1}^{K}\zeta_{2i}(t_{2i}^{\mathrm{A}}+t_{2i}^{\mathrm{B}})+\frac{y}{2K}\sum_{i=1}^{2K}S_i^{\mathrm{A}}S_i^{\mathrm{B}}+t_i^{\mathrm{A}}t_i^{\mathrm{B}}\right)$$
$$\times\left[\sum_{\mathrm{A,B}\in N_K}\exp\left(\frac{\beta}{\sqrt{K}}\sum_{i=1}^{K}\zeta_{2i}(t_{2i}^{\mathrm{A}}+t_{2i}^{\mathrm{B}})\right)\right]^{-1}. \tag{2.6}$$

Given any two states A and B it is natural to introduce the overlapping

$$q^{\mathrm{AB}}=\frac{1}{2K}\sum_{i=1}^{2K}S_i^{\mathrm{A}}S_i^{\mathrm{B}}+t_i^{\mathrm{A}}t_i^{\mathrm{B}} \tag{2.7}$$

and the distance

$$[d(A,B)]^2=\frac{1}{2K}\sum_{i=1}^{2K}(S_i^{\mathrm{A}}-S_i^{\mathrm{B}})^2+(t_i^{\mathrm{A}}-t_i^{\mathrm{B}})^2$$
$$=2(m^{*2}-q^{\mathrm{AB}}).$$

In our model, the probability $P_\zeta^K(q)$ that a pair of states has an overlapping greater than $q$ can be explicitly evaluated in term of the weights appearing in equation (2.6):

$$P_\zeta^K(q)=\sum_{\mathrm{A,B}\in N_K}\exp\left(\frac{\beta}{\sqrt{K}}\sum_{i=1}^{K}\zeta_{2i}(t_{2i}^{\mathrm{A}}+t_{2i}^{\mathrm{B}})\right)\theta\left(\frac{1}{2K}\sum_{i=1}^{2K}(S_i^{\mathrm{A}}S_i^{\mathrm{B}}+t_i^{\mathrm{A}}t_i^{\mathrm{B}})q\right)$$
$$\times\left[\sum_{\mathrm{A,B}\in N_K}\exp\left(\frac{\beta}{\sqrt{K}}\sum_{i=1}^{K}\zeta_{2i}(t_{2i}^{\mathrm{A}}+t_{2i}^{\mathrm{B}})\right)\right]^{-1} \tag{2.8}$$

where

$$\theta(x)=1 \qquad \text{if } x\geqslant 0$$
$$\theta(x)=0 \qquad \text{if } x<0.$$

In § 3 we show that in the limit $K\to\infty$ the set of the pure equilibrium states exhibits, almost surely, the hierarchical structure of an ultrametric space (Bourbaki 1966), i.e.

if we take any three states and we compute the three distances, at least two of which are equal.

(d) $P_\zeta^K(q)$ is the quantity proposed by Parisi (1983) as order parameter for spin glasses. Notice that, due to dependence on the sample $\{\zeta_{2i}\}$ (see (2.8)), $P^K(q)$ is a probability distribution-valued random variable.

## 3. Calculation of the free energy and determination of the pure equilibrium states

We will show that the free energy exists $\mu_1$ almost everywhere and does not depend on the actual values of the $\xi_\alpha$, $\eta_\alpha$ and $\zeta_{2i}$.

Let us start with $J_1 = 0$. In this case we have $2K$ independent blocks $\Omega_1 \ldots \Omega_{2K}$. $|\Omega_i| = M$ and the partition function $Z_N$ is given by

$$Z_N = \prod_{i=1}^{2K} Z_i$$

$$Z_i = \sum_{\{\sigma_{\Omega_i}\}} \exp \beta M J_0\left(\frac{S_i^2}{2} + \frac{t_i^2}{2}\right) + \beta\delta_i t_i \tag{3.1}$$

where

$$\delta_i = \frac{1+(-1)^i}{2}\frac{1}{\sqrt{K}}\zeta_i.$$

It is convenient to express the Hamiltonian

$$H_0(\sigma_{\Omega_i}) = -MJ_0(\tfrac{1}{2}S_i^2 + \tfrac{1}{2}t_i^2) - \delta_i t_i \tag{3.2}$$

in terms of the variables $\rho_\alpha$ and $\lambda_\alpha$ given by

$$\begin{aligned}\rho_\alpha &= \tfrac{1}{2}(\eta_\alpha + \xi_\alpha)\sigma_\alpha \\ \lambda_\alpha &= \tfrac{1}{2}(\eta_\alpha - \xi_\alpha)\sigma_\alpha.\end{aligned} \tag{3.3}$$

From (3.1), (3.2) and (3.3) we have

$$H_0(\sigma_{\Omega_i}) = -J_0\frac{1}{M}\left[\left(\sum_{\alpha\in\Omega_i^+}\rho_\alpha\right)^2 + \left(\sum_{\alpha\in\Omega_i^-}\lambda_\alpha\right)^2\right] - \delta_i\frac{1}{M}\sum_{\alpha\in\Omega_i}(\rho_\alpha - \lambda_\alpha) \tag{3.4}$$

where

$$\Omega_i^+ \equiv \{\alpha \in [iM+1, (i+1)M] / \xi_\alpha = \eta_\alpha\}$$

$$\Omega_i^- \equiv \{\alpha \in [iM+1, (i+1)M] / \xi_\alpha = -\eta_\alpha\}.$$

Calling

$$r_i = \frac{1}{M_i^+}\sum_{\alpha\in\Omega_i^+}\rho_\alpha \qquad l_i = \frac{1}{M_i^-}\sum_{\alpha\in\Omega_i^-}\lambda_\alpha$$

where $M_i^+ = |\Omega_i^+|$, $M_i^- = |\Omega_i^-|$, we get

$$S_i = \frac{M_i^+}{M} r_i + \frac{M_i^-}{M} l_i \qquad t_i = \frac{M_i^+}{M} r_i - \frac{M_i^-}{M} l_i \tag{3.5}$$

and

$$H_0(\sigma_{\Omega_i}) = -J_0 M\left[\left(\frac{M_i^+}{M}\right)^2 r_i^2 + \left(\frac{M_i^-}{M}\right)^2 l_i^2\right] - \delta_i r_i - \delta_i l_i.$$

The Hamiltonian $H_0(\sigma_{\Omega_i})$ describes two independent Curie–Weiss systems in $\Omega_i^+\Omega_i^-$ so that

$$Z_i = \sum_{r_i} \exp\left[\beta J_0 M\left(\frac{M_i^+}{M}\right)^2 r_i^2 + \beta\delta_i r_i + \log\binom{M_i^+}{\frac{1}{2}(1+r_i)M_i^+}\right]$$
$$\times \sum_{l_i} \exp\left[\beta J_0 M\left(\frac{M_i^-}{M}\right)^2 l_i^2 + \beta\delta_i l_i + \log\binom{M_i^-}{\frac{1}{2}(1+l_i)M_i^-}\right]. \tag{3.6}$$

Since, by the strong law of large numbers, $M_i^-/n \to \frac{1}{2}$, $M_i^+/n \to \frac{1}{2}$, almost surely when $M$ goes to infinity, the usual Laplace method leads to

$$\lim_{M\to\infty} \frac{1}{M} \log Z_i = \frac{1}{2} \max_{r,l} \left(\left(\frac{r^2}{2} + \frac{l^2}{2}\right) - \mathscr{E}(r) - \mathscr{E}(l)\right)$$
$$\equiv \max_{S,t} \left(\frac{S^2}{2} + \frac{t^2}{2} - \frac{1}{2}\mathscr{E}\left(\frac{s+t}{2}\right) - \frac{1}{2}\mathscr{E}\left(\frac{S-t}{2}\right)\right) \tag{3.7}$$

for $\mathscr{E}(x)$ cf equation (2.5).

It is an easy matter to check that for $\beta \leqslant \beta_c \equiv 1/J_0$

$$r_i = l_i = 0 \tag{3.8}$$

(which is equivalent to $S_i = t_i = 0$) is the only maximum. Meanwhile for $\beta > \beta_c$ the absolute maxima are four and expressing them in terms of the old variables we get

$$\begin{array}{ll} S_i = m^* & t_i = 0 \\ S_i = -m^* & t_i = 0 \\ S_i = 0 & t_i = m^* \\ S_i = 0 & t_i = -m^* \end{array} \tag{3.9}$$

where $\pm m^*$ are the non-zero solution of the equation

$$m^* = \tanh \beta J_0 m^*. \tag{3.10}$$

The free energy $F_i(\beta)$ of the $i$th block is given by

$$-\beta F_i(\beta) = \tfrac{1}{2}\beta J_0 m^{*2} - \mathscr{E}(m^*). \tag{3.11}$$

In the case $J_1 > 0$ a new interaction among the blocks is switched on and the free energy $F(\beta)$ becomes

$$-\beta F(\beta) = \max_{s,t} \mathscr{F}(s, t, \beta) \tag{3.12}$$

where

$$\mathscr{F}(s, t, \beta) = \frac{1}{2K} A + \frac{1}{2K} \sum_{i=1}^{2K} \tfrac{1}{2}(S_i^2 + t_i^2)\beta J_0 - \tfrac{1}{2}\mathscr{E}\tfrac{1}{2}(S_i + t_i) - \tfrac{1}{2}\mathscr{E}\tfrac{1}{2}(S_i - t_i) \tag{3.13}$$

with

$$A = \sum_{l=1}^{K} \sum_{J=2l+1}^{2K} [(t_{2l} - t_{2l-2})^4 + S_{2l-2}^4] t_J^4 \frac{1}{|2l - J|^{1+\varepsilon}}. \tag{3.14}$$

If we remark that $A \geqslant 0$ it is easy to convince ourselves that, for any given $K$, all the minima of the total free energy in the limit $M \to \infty$ are given by the solutions of the equation $A = 0$ that satisfy the constraints given by equation (3.8) for $\beta \leqslant \beta_c$ or by equation (3.9) for $\beta > \beta_c$. It turns out that for $\beta \geqslant \beta_c$, the absolute minima have the following form for $J = 1, 2, \ldots, K$:

$$\left.\begin{array}{ll} S_{2i} = 0 & t_{2i} = m^* \\ S_{2i+1} = \pm m^* \text{ or } 0 & t_{2i+1} = \pm m^* \text{ or } 0 \end{array}\right\} \quad \text{if } 1 \leqslant i \leqslant J$$

$$S_i = \pm m, \qquad t_i = 0 \qquad \text{if } 2J + 1 \leqslant i \leqslant 2K$$

or the analogous ones with $t_{2i} = -m^*$ if $1 \leqslant i \leqslant J$ (see figure 1).

Meanwhile, given a choice of $J$, the values of $S_i$ and $t_i$ are fixed for all blocks $2i$, $1 \leqslant i \leqslant j$. Their values can be chosen in $4^J$ ways for the odd blocks $2i - 1$ ($i \leqslant J$) and in $2^{2K-2J}$ ways for the remaining blocks.

Therefore $\forall \beta > \beta_c$ the number of equilibrium states is $2K\ 2^{2K}$ but the associated free energy is always

$$-\beta F(\beta) = \tfrac{1}{2}\beta J_0 m^{*2} - \mathscr{E}(m^*). \tag{3.15}$$

If we consider the Hessian of $\mathscr{F}(s, t, \beta)$ evaluated at a point satisfying (3.14) we immediately see that the contribution coming from the $A$ terms are identically zero, so that the behaviour around the minima is quadratic as in the case of the uncoupled systems, so that for any finite $K$ in the limit $M$ going to infinity the equilibrium measures are of the form

$$\prod_{i=1}^{2K} \delta(t_i - t_i^{\mathrm{A}}(J))\delta(S_i - S_i^{\mathrm{A}}(J)) \tag{3.16}$$

where $S_i(J)$ and $t_i(J)$ have to be chosen following a pattern consistent with (3.14).

All the previous equilibrium states can be collected in classes, that we will call instantons, with the following rule: we give an integer $J \in [1, K]$ (the length of the instanton) and the sign of the $t_i$ on the even blocks belonging to $[1, 2J]$ and consider all the equilibrium states with these properties. Then an instanton is characterised by an integer $J$ and a variable with values in $\{-m^*, +m^*\}$ (see figure 1).

## 4. The overlapping between two equilibrium states: the Parisi order parameter

For any given pair of pure states A and B, calling $J_{\rm A}$ and $J_{\rm B}$ the length of the associated instantons, from the result of the previous section (2.6) and (2.7), we obtain

$$q^{\rm AB}=\frac{1}{2K}\sum_{i=1}^{2K} S_i^{\rm A}S_i^{\rm B}+t_i^{\rm A}t_i^{\rm B}$$

$$=\frac{\varepsilon_{\rm A}\varepsilon_{\rm B}}{2K}\min(J_{\rm A},J_{\rm B})+\frac{1}{2K}\sum_{i=1}^{\max(J_{\rm A},J_{\rm B})} S_{2i-1}^{\rm A}S_{2i-1}^{\rm B}+\frac{1}{2K}\sum_{i=1}^{\min(J_{\rm A},J_{\rm B})} t_{2i-1}^{\rm A}t_{2i-1}^{\rm B}$$

$$+\frac{1}{2K}\sum_{i=2\max(J_{\rm A},J_{\rm B})}^{2K} S_i^{\rm A}S_i^{\rm B}. \tag{4.1}$$

From equation (2.6) we get also that the three sums on the right-hand side of equation (4.1) are sums of independent random variables with mean zero. By the law of large number we get for $J_{\rm A}=ak$ and $J_{\rm B}=bk$

$$\lim_{K\to\infty} q^{\rm AB}=\min(a,b)\frac{\varepsilon_{\rm A}\varepsilon_{\rm B}}{2}. \tag{4.2}$$

Furthermore if we remark that all the states associated with the instanton $J$ have the same weight:

$$f_\zeta^\varepsilon(J)=\exp\left(\varepsilon\beta m^*\frac{1}{\sqrt{K}}\sum_{i=1}^{J}\zeta_{2i}\right)$$

$$\times\left[\sum_{S=1}^{K}\exp\left(\beta m^*\frac{1}{\sqrt{K}}\sum_{i=1}^{s}\zeta_{2i}\right)+\exp\left(-\beta m^*\frac{1}{\sqrt{K}}\sum_{i=1}^{s}\zeta_{2i}\right)\right]^{-1}. \tag{4.3}$$

Given $q$, the probability of having an overlap bigger than $q$ is

$$P_\zeta^K(q)=\left(\sum_{J=[2qK]+1}^{K} f_\zeta^+(J)\right)^2+\left(\sum_{J=[2qK]+1}^{K} f_\zeta^-(J)\right)^2 \tag{4.4}$$

for

$$0<q<\tfrac{1}{2}$$

$$P_\zeta^K(q)=1-2\left(\sum_{J=[2|q|K]+1}^{K} f_\zeta^+(J)\right)\left(\sum_{J=[2|q|K]+1}^{K} f_\zeta^-(J)\right) \tag{4.5}$$

for

$$-\tfrac{1}{2}<q<0$$

and zero otherwise.

Looking at formula (4.3), it is easy to convince oneself that $P_\zeta^K(q)$ is a non-self-averaging quantity for any fixed $K$.

The question that naturally arises is if such a feature is preserved in the $K\to\infty$ limit.

Given $J_A=[\alpha K]$ and $J_B=[\beta K]$, it is easy to check that

$$\mathbb{E}\mu_2\left(\frac{1}{\sqrt{K}}\sum_{i=1}^{[\alpha K]}\zeta_{2i}\frac{1}{\sqrt{K}}\sum_{i=1}^{[\beta K]}\zeta_{2i}\right)\xrightarrow[K\to\infty]{}\inf(\alpha,\beta)$$

i.e. the covariance of the Brownian motion. This suggests that, in some way,

$$\sum_{J=[2|q|K]+1}^{K} f_\zeta^\varepsilon(J)\to\int_{2|q|}^{1}\exp(\varepsilon\beta m^* W(y))\,\mathrm{d}y \times\left(\int_0^1 2\cosh(\beta m^* W(y))\,\mathrm{d}y\right)^{-1}\equiv f_W^\varepsilon(|q|,\beta m^+) \tag{4.6}$$

where $W$ is a Brownian motion.

To make this statement precise, it is necessary to define the way the limit is taken and prove the uniform convergence with respect to $q$. To be consistent with our previous results that were all almost sure with respect to $\mu_1$ and $\mu_2$, we will require $\mu_2$ almost sure convergence. The proof, based on Skorohod's embedding scheme (Breiman 1968), is non-trivial but, since it is rather technical, we will skip it.

The result is the following: for any $K$ we can define, on a new probability space, a family of random variables $(\tilde{\zeta}_{2i})_{i=1}^K$ which have the same distribution as $(\zeta_{1i})_{i=1}^K$ together with a Brownian notion $W$ in such a way that if $K=K(I)$ is a subsequence increasing rapidly enough, (4.6) is true $\mu_2$ almost surely, uniformly with respect to $q$. Therefore $P^{(K)}(J)$ converges almost surely to the probability distribution-valued random variable

$$P_w(q)=[f_w^+(q,\beta m^*)]^2+[f_w^-(q,\beta m^*)]^2 \qquad \text{if } 0<q<\tfrac{1}{2}$$
$$P_w(q)=1-2[f_w^+(|q|,\beta m^*)][f_w^-(|q|,\beta m^*)] \qquad \text{if } -\tfrac{1}{2}<q<0$$

and zero otherwise. This is a rigorous way to describe a non-self-averaging effect and we think it is of some interest to be able to illustrate, in a simple model, problems and strategies that should be relevant for any random system.

## 5. Comments and remarks

*Remark 1.* It is natural to ask how much of the previous results are still valid if we introduce an external, possibly small, magnetic field $h$. Meanwhile the interaction among blocks stays the same and the self-interaction inside each block becomes

$$H_i=-MJ_0\left(\frac{S_i^2}{2}+\frac{t_i^2}{2}\right)-\delta_i t_i-Mhm_i.$$

Following the procedure of § 3, each block splits into two ferromagnetic sub-blocks with a random external field and the equation for the extrema of the free energy associated with each sub-block are of the form

$$a_1+a_2=\tfrac{1}{4}\{\tanh[2\beta(a_1+a_2)+\beta h]+\tanh[2\beta(a_1+a_2)-\beta h]\}$$
$$a_1=\tfrac{1}{4}\tanh[2\beta(a_1+a_2)+\beta h]$$

where $a_1=\frac{1}{2}(r+m)$, $a_2=\frac{1}{2}(r-m)$.

These equations have already been derived and discussed in similar (Luttinger 1976) or slightly different contexts (Gorter and Peski-Tinbergen 1956, Kincaid and Cohen 1975) and a detailed study, both graphical and numerical, is performed in

Gorter and Peski-Tinbergen (1956) and results therein are summarised by Kincaid and Cohen (1975).

From this analysis it emerges that while for $\beta < \beta_c$ and arbitrary $h$ the only solution is $a_1 = a_2 = 0$, for $\beta > \beta_c$ the phase diagram in the $(\beta, h)$ plane becomes quite complicated with two and even three coexisting phases in each sub-block. However the limit $h \to 0$ is still simple and it is easy to show that $\forall \beta > \beta_c$ and suitably small $h$, the equilibrium states are very similar to those of the $h = 0$ case.

*Remark 2.* This class of mean field random models involving only global variables suitably normalised (e.g. $S = \Sigma\, \xi_\alpha \sigma_\alpha / M$) can be studied following a very general procedure based on large deviation theory (van Hemmen 1982). We think it is worth pointing out that the procedure we use in § 3 is less general but has two advantages.

(i) It is more transparent from a physical point of view (for every sample of the $\xi_\alpha$ and $\eta_\alpha$ the blocks split into two smaller blocks with ferromagnetic interactions only).

(ii) It allows us, via the Stirling formula, to get explicitly the non-leading term in $M$. Those terms, not relevant for the evaluation of the free energy and the determination of the extrema, are necessary to perform the estimate made in the appendix.

*Remark 3.* Each equilibrium state of our system is associated with an 'instanton' and all instantons start at $i = 0$ (cf equation (3.14)). To avoid this unpleasant feature, that singles out a particular site, our system can be thought of as a circle and the origin of the instantons as another random variable.

Also the feature that only even blocks have to be 'aligned' can be avoided. If we consider two sequences $\{C_n\}_{n\in\mathbb{N}}$ and $\{D_n\}_{n\in\mathbb{N}}$ of sets of blocks:

$$|C_n| = |D_n| = n$$

$$\left(\bigcup_n C_n\right) \cap \left(\bigcup_n D_n\right) = \varnothing$$

$$C_n \subset C_{n'} \qquad D_n \subset D_{n'} \qquad \forall n < n'$$

it is easy to construct an interaction similar to that in (2.2) where the role played by odd and even blocks with index less than $2n$ is now played by the blocks belonging to $C_n$ and $D_n$.

These are all 'cosmetic' improvements and they cannot hide the real drawback of this model. The randomness associated with the $\xi_\alpha$ and $\eta_\alpha$ is too weak to play the role of the $J_{ij}$ of the SK model and we have to introduce an interaction on a different scale length to induce the hierarchical structure of the states. But as we have pointed out in the introduction our aim is not that of proposing a new candidate for the Hamiltonian of spin glass systems but simply to exhibit a statistical mechanical model where it is possible to test, by standard methods, ideas and procedures still ill defined in more complex models.

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**Appendix**

In order to compute

$$\lim_{M,K\to\infty} g_{M,K}(y) = \lim_{M,K\to\infty} \sum_{\{\sigma_\Omega,\sigma\Omega'\}} \exp\{-\beta[H(\sigma_\Omega)+H(\sigma'_\Omega)]\} + \frac{y}{2KM}\sum_{\alpha=1}^{2KM} \sigma_\alpha\sigma'_\alpha \left(\sum_{\{\sigma_\Omega,\sigma'_\Omega\}} \exp\{-\beta[H(\sigma_\Omega)+H(\sigma'_\Omega)]\}\right)^{-1} \tag{A1}$$

we define

$$W_i^+ = \frac{1}{M_i^+}\sum_{\alpha\in\Omega_i^+} \sigma_\alpha\sigma'_\alpha \qquad W_i^- = \frac{1}{M_i^-}\sum_{\alpha\in\Omega_i^+} \sigma_\alpha\sigma'_\alpha \tag{A2}$$

$$\Sigma_i^+ \equiv \{1,+1\}^{M_i^+} \qquad \Sigma_i^- \equiv \{-1,+1\}^{M_i^-} \tag{A3}$$

and recall from § 3 that

$$S_i = (M_i^+/M)r_i + (M_i^-/M)l_i \tag{A4}$$

$$t_i = (M_i^+/M)r_i - (M_i^-/M)l_i \tag{A5}$$

so that $g_{M,K}$ can be written

$$\sum_{\substack{r,r'\\l,l'}} \exp\{-\beta[H(r,l)+H(r',l')]\} \prod_{i=1}^{2K} \left(\sum_{\sigma,\sigma'}^{\substack{r_i r'_i\\ l_i l'_i}} \exp\frac{y}{2K}\left(\frac{M_i^+}{M}W_i^+ + \frac{M_i^-}{M}W_i^{-1}\right)\right.$$
$$\times\left(\sum_{\substack{r,r'\\l,l'}} \exp\{-\beta[H(r,l)+H(r',l')]\} \prod_{i=2}^{2K}\left(\sum_{\sigma,\sigma'}^{\substack{r_i r'_i\\ l_i l'_i}} 1\right)\right]^{-1} \tag{A6}$$

where $\sum_{\sigma,\sigma'}^{\substack{r_i r'_i\\ l_i l'_i}}$ runs over all configurations $\{\sigma,\sigma'\}$ such that

$$\begin{aligned}
&\frac{1}{M_i^+}\sum_{\alpha\in\Omega_i^+} \sigma_\alpha(\xi_{(\alpha+\eta\alpha)/2}) = r_i\\
&\frac{1}{M_i^-}\sum_{\alpha\in\Omega_i^-} \sigma_\alpha(\xi_{(\alpha-\eta\alpha)/2}) = l_i\\
&\frac{1}{M_i^+}\sum_{\alpha\in\Omega_i^+} \sigma'_\alpha(\xi_{(\alpha+\eta\alpha)/2}) = r'_i\\
&\frac{1}{M_i^-}\sum_{\alpha\in\Omega_i^-} \sigma'_\alpha(\xi_{(\alpha-\eta\alpha)/2}) = l'_i.
\end{aligned} \tag{A7}$$

We consider the following random variable

$$f(r_i, r'_i, l_i, l'_i)$$
$$= \left[\sum_{\sigma,\sigma'}^{\substack{r_i r'_i\\ l_i l'_i}} \exp\left(\frac{y}{2KM}(M_i^+W_i^+ + M_i^-W_i^-)\right)\right]\left(\sum_{\sigma,\sigma'}^{\substack{r_i r'_i\\ l_i l'_i}} 1\right)^{-1} \tag{A8}$$

$$= \left[\sum_{\sigma,\sigma'\in\Sigma_i^+}^{t_i r_i} \exp\left(\frac{y}{2K}\frac{M_i^+}{M}W_i^+\right)\right]\left[\sum_{\sigma,\sigma'\in\Sigma_i^-}^{l_i l_i} \left(\exp\frac{y}{2K}\frac{M_i^-}{M}W_i^-\right)\right]$$
$$\times\left(\sum_{\sigma,\sigma'\in\Sigma_i^+} 1\right)^{-1}\left(\sum_{\sigma,\sigma'\in\Sigma_i^-} 1\right)^{-1}. \tag{A9}$$

Let us define if $0<\delta<\frac{1}{2}$

$$\Xi_i = \left[ \xi_\alpha, \eta_\alpha, \alpha \in \Omega_i \left(\frac{M}{2}\right)^{-1} - \left(\frac{M}{2}\right)^{1/2+\delta} \leqslant M_i^+ \leqslant \frac{M}{2} + \left(\frac{M}{2}\right)^{1/2+\delta} \right]$$

and

$$\Xi(M, K) = \bigcup_{i=1}^{2K} \Xi_i.$$

We want to study

$$f(r_i, r_i', \Sigma_i^+) = \left[ \sum_{\sigma,\sigma' \in \Sigma_i^+}^{r_i r_i'} \exp\left(\frac{y}{2K} \frac{M_i^+}{M} W_i^2\right)\right] \left( \sum_{\sigma,\sigma' \in \Sigma_i^+}^{r_i r_i'} 1 \right)^{-1} \tag{A10}$$

for $M \to \infty$.

*Lemma*

$$f(r_i r_i', \Sigma_i^+) = \exp\left(\frac{y}{2K} \frac{1}{2} r_i r_i'\right) + \mathrm{o}(M). \tag{A11}$$

*Proof.* Using

$$\begin{aligned} h(r, r') &\equiv \sum_{\sigma,\sigma' \in \Sigma_i^+}^{r_i r_i'} \exp\left( y \sum_{\alpha \in \Omega_i^+} \frac{\sigma_\alpha \sigma'_\alpha}{M_i^+} \right) \\ &= \sum_{\sigma,\sigma' \in \Sigma_i^+} \delta\left( \sum_{\alpha \in \Omega_i^+} \frac{\sigma_\alpha}{M_i^+} - r_i \right) \delta\left( \sum_{\alpha \in \Omega_i^+} \frac{\sigma'_\alpha}{M_i^+} - r_i' \right) \exp\left( \frac{y}{M_i^+} \sum_{\alpha \in \Omega_i^+} \sigma_\alpha \sigma'_\alpha \right) \end{aligned} \tag{A12}$$

and

$$\delta\left( \sum_{\alpha \in \Omega_i^+} \frac{\sigma_\alpha}{M_i^+} - r_i \right) = \sum_{K=-\infty}^{+\infty} \exp\left[ \mathrm{i}K \left( \sum_{\alpha \in \Omega_i^+} \frac{\sigma_\alpha}{M_i^+} - r_i \right)\right].$$

We obtain

$$\begin{aligned} h(r_i, r_i') = \sum_{K=-\infty}^{+\infty} \sum_{K'=-\infty}^{+\infty} \exp[-i(Kr_i + K'r_i)] \sum_{\sigma,\sigma'} \prod_{\alpha \in \Omega_i^+} \\ \times \left[ \exp\left( \mathrm{i}K \frac{\sigma_\alpha}{M_i^+} + \mathrm{i}K' \frac{\sigma'_\alpha}{M_i^+} - y \frac{\sigma_\alpha \sigma'_\alpha}{M_i^+} \right)\right]. \end{aligned} \tag{A13}$$

Using the multinomial theorem and resumming over the indices $K$ and $K'$ we get

$$h(r_i, r_i') = \sum_{n_i}^{*} \frac{(M_i^+)!}{n_1! n_2! n_3! n_4!} \exp\left( \frac{y}{M_i^+} [n_1 + n_2 - n_3 - n_4] \right) \tag{A14}$$

where $\Sigma^*$ runs over all families $(n_1, n_2, n_3, n_4)$ of positive integers with the constraints

$$\begin{aligned} n_1 + n_2 + n_3 + n_4 &= M_i^+ \\ n_1 - n_2 + n_3 - n_4 &= M_i^+ r_i \\ n_1 - n_2 - n_3 - n_4 &= M_i^+ r_i'. \end{aligned} \tag{A15}$$

If we set $n_j = \alpha_i M_i^+$ it is not difficult to check by the Stirling formula that, under the previous constraints, $(M_i^+)!/n_1!n_2!n_3!n_4!$ is a maximum if

$$\begin{aligned}\alpha_1 &= \tfrac{1}{4}+\tfrac{1}{4}r_i+\tfrac{1}{4}r_ir_i'+\tfrac{1}{4}r_i' \\ \alpha_2 &= \tfrac{1}{4}-\tfrac{1}{4}r_i+\tfrac{1}{4}r_ir_i'-\tfrac{1}{4}r_i' \\ \alpha_3 &= \tfrac{1}{4}+\tfrac{1}{4}r_i-\tfrac{1}{4}r_ir_i'-\tfrac{1}{4}r_i' \\ \alpha_4 &= \tfrac{1}{4}-\tfrac{1}{4}r_i-\tfrac{1}{4}r_ir_i'+\tfrac{1}{4}r_i\end{aligned} \tag{A16}$$

and that this maximum is equal to

$$\exp(-M_i^+(-\log 4+\mathscr{E}(r_i)+\mathscr{E}(r_i')) \times\exp(-\tfrac{3}{2}\log M_i^+-\tfrac{1}{2}\log(1-r_i^2)-\tfrac{1}{2}\log(1-r_i'^2)+\mathrm{o}(M)) \tag{A17}$$

from which we get the result.

If we consider

$$g_{M,K}=\left[\sum_{\substack{r,r'\\l,l'}}\exp(-\beta[H(\boldsymbol{r},\boldsymbol{l})+H(\boldsymbol{r}',\boldsymbol{l}')])\prod_{i=1}^{2K}\left(\sum_{\sigma,\sigma'\in\Sigma_i^+}^{r_ir_i'}1\right)\times\left(\sum_{\sigma,\sigma'\in\Sigma_i^-}^{l_il_i'}1\right)f(r_i,r_i'\Sigma_i^+)f(l_i,l_i'\Sigma)\right]$$
$$\times\left[\sum_{\substack{r,r'\\l,l'}}\exp(-\beta[H(\boldsymbol{r},\boldsymbol{l})+H(\boldsymbol{r}',\boldsymbol{l}')])\prod_{i=1}^{2K}\left(\sum_{\sigma,\sigma'\in\Sigma_i^+}^{r_ir_i'}1\right)\left(\sum_{\sigma,\sigma'\in\Sigma_i^-}^{l_il_i'}1\right)\right]^{-1} \tag{A18}$$

the term $\exp(-\Sigma_{i=1}^{2K}(M_i^+\log 4-\frac{3}{2}\log M_i^++M_i^-\log 4-\frac{3}{2}\log M_i^-)$ which appears in $\Pi_{i=1}^{2K}(\Sigma\,1)(\Sigma\,1)$ in the numerator factorises and is cancelled by the same term in the denominator.

On the other hand the sum $\Sigma_{rr'll'}$ can be replaced by an integral over $[-1,+1]^{2K}$ and the main contribution of this integral comes from the neighbourhood, say balls of radius $\rho$, of the minimum defined by equation (3.14).

If $\rho^3M$ goes to zero when $M$ goes to infinity, it can be checked, by using $\mu_1(\lim_M \Xi(M,K))=1$ and the usual Laplace method that the contribution of a neighbourhood of $(\boldsymbol{s}^{\mathrm{A}},\boldsymbol{t}^{\mathrm{A}},\boldsymbol{s}'^{\mathrm{B}},\boldsymbol{t}'^{\mathrm{B}})$ is $\mu_1$ almost everywhere

$$\exp(-M(F(s^{\mathrm{A}},t^{\mathrm{A}},s'^{\mathrm{B}},t'^{\mathrm{B}})))\exp\left(\frac{\beta}{\sqrt{K}}\sum_{i=1}^{K}\zeta_{2i}(t_{2i}^{\mathrm{A}}+t_{2i}'^{\mathrm{B}})\right)\exp\left(\frac{y}{2K}\sum_{i=1}^{2K}s_i^{\mathrm{A}}s_i'^{\mathrm{B}}+t_i^{\mathrm{A}}t_i'^{\mathrm{B}}\right)$$
$$\times\exp\left(-\frac{1}{2}\sum_{i=2}^{2K}\log(1-(s_i^{\mathrm{A}}+t_i^{\mathrm{A}})^2)(1-(s_i'^{\mathrm{B}}+t_i'^{\mathrm{B}})^2)\right)$$
$$\times[\det H(s^{\mathrm{A}},t^{\mathrm{A}},s'^{\mathrm{B}},t'^{B})]^{-1/2}M^{-K} \tag{A19}$$

where $H(s^*,t^*,s'^*,t'^*)$ is the Hessian of the function $F$. It can also be checked that the total contribution which comes from the integral outside these neighbourhoods does not exceed $2^K/n\rho$. Therefore if $M_\rho$ goes to infinity this contribution goes to zero. It is not difficult to see that $\det H(s^{\mathrm{A}},t^{\mathrm{A}},s'^{\mathrm{B}},t'^{\mathrm{B}})$ and $\Sigma_{i=1}^{2K}\log[1-(s^{\mathrm{A}}-t^{\mathrm{A}})^2][1+(s^{\mathrm{B}}+t^{\mathrm{B}})^2]$ has the same value over all the minima defined by equation (3.14) so that it will factor out; therefore when $M$ goes to infinity we get

$$\lim_{M\to\infty} g_{M,K}(y) = \left[ \sum^*_{A,B\in N_K} \exp\left( \frac{\beta}{\sqrt{K}} \sum_{i=1}^{K} \zeta_{2i}(t^{A}_{2i} + t^{B}_{2i}) \right) + \frac{y}{2K} \sum_{i=1}^{2K} s^{A}_i s^{B}_i + t^{A}_i t^{B}_i \right] \times \left[ \sum_{A,B\in N_\mu} \exp\left( \frac{\beta}{\sqrt{K}} \sum_{i=1}^{K} \zeta_{2i}(t^{A}_{2i} + t^{B}_{2i}) \right) \right]^{-1} . \tag{A20}$$

## References


Bourbaki N 1966 *Espaces Vectoriels Topologiques* (Paris: Dunod)
Breiman L 1968 *Probability* (Reading, MA: Addison Wesley)
Gorter G J and Peski-Tinbergen T Van 1956 *Physica* **22** 274
Kincaid G M and Cohen E G D 1975 *Phys. Rep.* **22** 58
Luttinger J M 1976 *Phys. Rev. Lett.* **37** 778
Mezard M, Paris G, Sourlas N, Toulouse G and Virasoro M 1984a *Phys. Rev. Lett.* **52** 1156
—— 1984b *J. Physique* **45** 843
Parisi G 1983 *Phys. Rev. Lett.* **50** 1946
Thompson C J 1972 *Mathematucal Statistical Mechanics* (Princeton, NJ: Princeton University Press)
van Hemmen J L 1982 *Phys. Rev. Lett.* **49** 409